Bifurcation Analysis and Optimal Control of a Fractional-Order Predator-Prey Model with Fear Effect and Harvesting

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[1]*Date of Receiving: 11 October 2026; Date of Acceptance: 09 November 2026; Date of Publication: 18 November 2026*

Abstract

In this paper, the dynamical behavior and optimal harvesting management of a Caputo fractional-order predator-prey model that includes anti-predator fear effect on the prey reproduction, Holling type-II functional response, and linear harvesting of both species are discussed. The use of fractional calculus in the Caputo sense represents the memory and hereditary characteristics of ecological dynamics. The existence, uniqueness, positivity and boundedness of the solutions in the positive quadrant are verified. The entries of the Jacobian at the interior coexistence equilibrium are computed, and local asymptotic stability conditions for all biologically feasible equilibria are found using Matignon's stability criterion. Another analytical critical threshold α^* of Hopf bifurcation for fractional order α is also obtained and the phase-margin transversality condition is verified. The problem is formulated as a fractional optimal control problem and solved using Pontryagin's Maximum Principle, the concept is to balance harvesting revenue with ecological sustainability. The numerical simulations carried out by the generalized Adams-Bashforth-Moulton predictor-corrector method are in agreement with the analytical results.

Keywords: *Fractional-order dynamical system; Predator-prey model; Fear effect; Holling type-II response; Hopf bifurcation; Optimal control; Caputo derivative.*

1. Introduction

The mathematical modelling of predator-prey interactions continues to be a fundamental cornerstone of theoretical ecology, and of applied dynamical systems. The classical models based on Lotka-Volterra dynamics make the assumption that the regulation of prey populations is primarily via direct predation. Empirical and field studies such as those of Zanette et al. [1] and Wang et al. [2] indicate, however, that perceived predation risk alone can cause chronic stress responses in prey, which can lead to decreased reproductive rates and changes in foraging and habitat-use behavior without the animals actually being killed.

The classical integer-order ordinary differential equations (ODEs) are local operators and do not explicitly contain historical memory or hereditary characteristics like learned predator avoidance, physiological adaptation across generations. Systems with memory can be modeled using fractional-order differential equations (FDEs), which are obtained in a natural way [3, 4]. The Caputo fractional derivative is the most popular one of the existing formulations in ecological modeling since it can handle standard initial conditions that have direct biological interpretations.

If interacting species are caught commercially or regulated, further economic yield/persistence trade-offs arise. Too much or too early harvesting can disrupt community relationships or lead to the extinction of sensitive species, while

[1] How to cite the article: Barhoom N.S.S..; (November 2026); Bifurcation Analysis and Optimal Control of a Fractional-Order Predator-Prey Model with Fear Effect and Harvesting; *International Journal of Universal Science and Engineering*; Vol 13, 190-196

optimal harvesting based on time can lead to sustainable harvesting of resources [5]. Recently fractional-order predator-prey models with fear, group defense and/or harvesting were studied [10, 11, 12] but a combined analysis of fractional-order predator-prey models with fear, Holling type-II predation, dual-species linear harvesting, explicit Hopf bifurcation thresholds in α, and fractional optimal harvesting control would be worthy of a unified study. This paper aims to fill that void.

2. Preliminaries and Mathematical Model Formulation

Some basic definitions and results in fractional calculus that are used in our analysis are recalled.

Definition 1 (Caputo Fractional Derivative). For $\alpha \in (0,1]$ and a continuously differentiable function $f \in C^1([t_0,\infty),\mathbb{R})$, the Caputo fractional derivative of order α is defined by [3]:

$$ {}^{c}D_{t_0}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} (t-\tau)^{-\alpha} f'(\tau)\, d\tau \qquad (1)$$

As $\alpha \to 1$, the Caputo operator converges to the classical first-order derivative df/dt.

Lemma 1 (Generalized Mean Value Theorem [6]). Suppose $f(t) \in C[a,b]$ and ${}^{c}D_{a}^{\alpha} f(t) \in C(a,b]$ for $0 < \alpha \leq 1$. Then for every $t \in (a,b]$, there exists $\xi \in (a,t)$ such that $f(t) = f(a) + \frac{1}{\Gamma(\alpha)} ({}^{c}D_{a}^{\alpha} f)(\xi)(t-a)^{\alpha}$. In particular, if ${}^{c}D_{a}^{\alpha} f(t) \geq 0$ for all $t \in (a,b)$, then $f(t)$ is non-decreasing on $[a,b]$.

Incorporating the anti-predator fear response factor $\frac{1}{(1+ky)}$ into prey growth, a Holling type-II functional response $\frac{cx}{1+mx}$, and linear harvesting efforts E_1, E_2 yields the Caputo fractional-order system:

$$\begin{aligned} {}^{c}D_{t}^{\alpha} x(t) &= \frac{rx}{1+ky} - d_1 - ax^2 - \frac{ecxy}{1+mx} - q_1 E_1 x \\ {}^{c}D_{t}^{\alpha} y(t) &= \frac{ecxy}{1+mx} - d_2 y - q_1 E_2 y \end{aligned} \qquad (2)$$

subject to strictly positive initial conditions $x(0) = x_0 > 0$ and $y(0) = y_0 > 0$. All model parameters are positive and listed in Table 1.

Table 1: Parameters and baseline numerical values for system (2).

Parameter	Biological Description	Baseline Value
α	Order of Caputo fractional derivative ($0 < \alpha \leq 1$)	0.72 – 0.96 (test cases)
r	Intrinsic growth rate of prey population	2.00
k	Level of anti-predator fear induced on prey	0.30
d_1, d_2	Natural mortality rates of prey and predator	0.10, 0.25
a	Intra-specific competition coefficient among prey	0.02
c	Maximum predation (capture) rate of predator	1.00
m	Half-saturation constant (Holling type-II response)	0.80
e	Biomass conversion efficiency ($0 < e < 1$)	0.70
q_1, q_2	Catchability coefficients of prey and predator	0.10, 0.10
E_1, E_2	Linear harvesting efforts on prey and predator	0.10, 0.10

3. Qualitative Analysis of the Model

3.1 Existence, Uniqueness, Non-negativity, and Uniform Boundedness

Theorem 1 (Well-Posedness). For any initial condition $(x_0, y_0) \in \mathbb{R}^2_+$, system (2) admits a unique global solution $(x(t), y(t)) \in \mathbb{R}^2_+$ for all $t \geq 0$ that remains non-negative and uniformly bounded.

Proof (Existence and Uniqueness). Let $X(t) = (x(t), y(t))^{\mathrm{T}}$ and denote the right-hand side of (2) by $F(X) = (F_1(X), F_2(X))^T$. On the compact set $\Omega = \{(x, y) \in \mathbb{R}_+{}^2 : max(|x|, |y|) \leq M\}$ for sufficiently large $M > 0, F(X)$ is continuously differentiable and thus satisfies a local Lipschitz condition on Ω with Lipschitz constant $L = max(L_1, L_2)$, where $L_1 = r + d_1 + 2aM + cM + q_1E_1 + ecM$ and $L_2 = rkM + cM + d_2 + q_2E_2 + ecM$. By the Picard-Lindelöf theorem for fractional differential equations [3], a unique local solution exists.

Proof (Non-negativity). Along the boundary axes, ${}^cD_t^\alpha x(t)\,|_{\{x=0\}} = 0$ and ${}^cD_t^\alpha y(t)|_{\{y=0\}} = 0$. By Lemma 1, trajectories starting in $\mathbb{R}^2_+$ cannot cross the coordinate axes, so $\mathbb{R}^2_+$ is positively invariant.

Proof (Uniform Boundedness). Define $W(t) = x(t) + (1/e)y(t)$. Taking the Caputo derivative along solutions of (2) gives ${}^cD_t^\alpha\, W(t) = \frac{rx}{(1+ky)} - (d_1 + q_1E_1)x - ax^2 - (\frac{(d_2+q_2E_2)}{e})y$. Setting $\eta = min(d_1 + q_1E_1, d_2 + q_2E_2) > 0$ and using $\frac{rx}{(1+ky)} \leq rx$ for $y \geq 0$ yields ${}^cD_t^\alpha\, W(t) + \eta\, W(t) \leq rx - ax^2 \leq r^2/(4a)$. Applying the fractional comparison principle gives $W(t) \leq (W(0) - r^2/(4a\eta))\, E_\alpha(-\eta\, t^\alpha) + r^2/(4a\eta)$, where $E_\alpha(\cdot)$ is the Mittag-Leffler function. Since $E_\alpha(-\eta t^\alpha) \to 0$ as $t \to \infty, lim\ sup_{\{t\to\infty\}} W(t) \leq r^2/(4a\eta)$. Hence solutions are uniformly bounded and extend globally for all $t \geq 0$. ■

3.2 Equilibrium Points and Local Asymptotic Stability

Setting ${}^cD_t^\alpha x(t) = 0$ and ${}^cD_t^\alpha y(t) = 0$ in system (2) gives the following equilibria:

1. Trivial equilibrium $P_0 = (0, 0)$, which always exists.

2. Predator-free equilibrium $P_1 = (\frac{(r - d_1 - q_1E_1)}{a}, 0)$, which is biologically feasible when $r > d_1 + q_1E_1$.

3. Interior coexistence equilibrium $P^* = (x^*, y^*)$. From the predator equation with $y^* > 0$, we obtain $x^* = \frac{(d_2+q_2E_2)}{(ec - m(d_2+q_2E_2))}$, which is positive provided $ec > m(d_2 + q_2E_2)$. Substituting $x^* =$ into the prey equilibrium equation $\frac{r}{(1+ky^*)} - d_1 - ax^* - \frac{cy^*}{(1+mx^*)} - q_1E_1 = 0$ and clearing denominators (using $1 + mx^* > 0$ and $1 + ky^* > 0$) leads to the quadratic equation $A\, y^2 + B\, y + C = 0$, where:

$$A = \frac{ck}{1+mx},\ B = \frac{c}{1+mx^*} + k(d_1 + ax^* + q_1E_1),\ C = d_1 + ax^* + q_1E_1 - r \qquad (3)$$

Because $A > 0$ and $B > 0$, Descartes' rule of signs implies a unique positive root $y^* > 0$ if and only if $C < 0$, i.e., $r > d_1 + ax^* + q_1E_1$.

Evaluating the Jacobian matrix of system (2) at $P^* = (x^*, y^*)$ and using the equilibrium relation $r/(1 + ky^*) - d_1 - q_1E_1 = ax^* + cy^*/(1 + mx^*)$ gives:

$$J(P^*) = [J^*_{11}, J^*_{12}; J^*_{21}, 0],\ J^*_{11} = -ax^* + \frac{cmx^*y^*}{(1+mx^*)^2}, J^*_{12} = -\frac{rkx^*}{(1+ky^*)^2} - \frac{cx^*}{(1+mx^*)}, J^*_{21} = \frac{ecy^*}{(1+mx^*)^2} \qquad (4)$$

Theorem 2 (Local Stability via Matignon's Criterion [7]). The coexistence equilibrium P^* is locally asymptotically stable if all eigenvalues $\lambda_{1,2}$ of $J(P^*)$ satisfy $|arg(\lambda_i)| > \alpha\pi/2$. Specifically, P^* is locally asymptotically stable if either: (i) $J^*_{11} \leq 0$, or (ii) $J^*_{11} > 0$, $(J^*_{11})^2 < 4\, Det(J^*)$, and $\alpha < \frac{2}{\pi} arctan\left(\frac{\sqrt{4Det(J^*) - (J^*_{11})^2}}{J^*_{11}}\right)$.

Proof. The characteristic equation is $\lambda^2 - Tr(J^*)\lambda + Det(J^*) = 0$, where $Tr(J^*) = J_{11}^*$ and $Det(J^*) = -J_{12}^* J_{21}^* > 0$. For Case (i), $J_{11}^* \leq 0$ and $Det(J^*) > 0$ $imply$ $|arg(\lambda_{1,2})| \geq \pi/2 > \alpha\pi/2$ for all $\alpha \in (0,1]$, so P^* is locally asymptotically stable by Matignon's theorem [7]. For Case (ii), $J_{11}^* > 0$ and $(J_{11}^*)^2 < 4\,Det(J^*)$ yield complex conjugate eigenvalues $\lambda^{1,2} = \mu \pm i\omega$ with $\mu = J_{11}^*/2 > 0$ and $\omega = \frac{\sqrt{4Det(J^*)-(J_{11}^*)^2}}{2} > 0$. Thus $|arg(\lambda_{1,2})| = arctan(\omega/\mu) > \alpha\pi/2$ holds whenever $\alpha < (2/\pi)\,arctan(\omega/\mu)$.

4. Hopf Bifurcation Analysis

In fractional-order systems, a Hopf bifurcation can occur when complex conjugate eigenvalues cross the Matignon stability boundary $|arg(\lambda)| = \alpha\pi/2$ as the fractional order α varies.

Theorem 3 (Existence of Hopf Bifurcation). Suppose P^* exists, $Det(J^*) > 0, J_{11}^* > 0$, and $(J_{11}^*)^2 < 4\,Det(J^*)$. Then system (2) undergoes a Hopf bifurcation around P^* when α crosses the critical value:

$$\alpha^* = \frac{2}{\pi} arctan\left(\frac{\sqrt{4Det(J^*)-(J_{11}^*)^2}}{J_{11}^*}\right) \qquad (5)$$

Proof. Define the phase-margin function $\Phi(\alpha) = |arg(\lambda_{1,2})| - \alpha\pi/2 = arctan\left(\frac{\sqrt{4Det(J^*)-(J_{11}^*)^2}}{J_{11}^*}\right) - \frac{\alpha\pi}{2}$. At $\alpha = \alpha^*$, we have $\Phi(\alpha^*) = 0$, so $\lambda_{1,2}$ lie on the boundary rays $arg(\lambda) = \pm\frac{\alpha^*\pi}{2}$. For $\alpha < \alpha^*, \Phi(\alpha) > 0$ (stable focus/spiral); for $\alpha > \alpha^*, \Phi(\alpha) < 0$ (unstable spiral). Moreover, $\frac{d\Phi}{d\alpha}|_{\{\alpha=\alpha*\}} = -\frac{\pi}{2} \neq 0$, satisfying the transversality condition and establishing the emergence of periodic oscillations as α crosses α*. ■

5. Fractional Optimal Control Problem

Consider time-dependent harvesting efforts $E_1(t)$ and $E_2(t)$ belonging to the admissible control set $U = \{(E_1(t), E_2(t)) : 0 \leq E_1(t) \leq E_1max, 0 \leq E_2(t) \leq E_2max, t \in [0, T_f]\}$. We formulate the net economic objective functional $J(E_1, E_2)$, the fractional Hamiltonian H, and the bounded optimal harvesting strategies $(E_1^*(t), E_2^*(t))$ via Pontryagin's Maximum Principle [8] as:

$$J(E_1, E_2) = \int_0^{\tau_1} \left[(p_1 q_1 x(t) - c_1)E_1 + (p_2 q_2 y(t) - c_2)E_2 - \frac{\omega_1}{2}E_1^2 - \frac{\omega_2}{2}E_2^2\right] dt,$$
$$\mathcal{H} = \left((p_1 q_1 x(t) - c_1)E_1 + (p_2 q_2 y(t) - c_2)E_2\right) - \frac{\omega_1}{2}E_1^2 - \frac{\omega_2}{2}E_2^2 + \xi_1(t)\,{}^cD_t^\alpha y \qquad (6)$$
$$E_1^* = min\left\{max\left\{0, \frac{(p_1 q_1 + \xi_1(t) q_1)x(t) - c_1}{w_1}\right\}, E_{1,max}\right\}, E_2^* = min\left\{max\left\{0, \frac{(p_2 q_2 + \xi_2(t) q_2)x(t) - c_2}{w_1}\right\}, E_{2,max}\right\}$$

where p_1, p_2 are unit biomass prices, c_1, c_2 are operational harvesting costs, $w_1, w_2 > 0$ are quadratic effort cost weights, and the co-state variables $\xi_1(t), \xi_2(t)$ satisfy the adjoint equations with terminal conditions $\xi_1(T_f) = \xi_2(T_f) = 0$.

6. Numerical Simulations and Discussion

We simulate system (2) using the generalized Adams-Bashforth-Moulton predictor-corrector scheme [9] with step size $h = 0.05$ and baseline parameters from Table 1 ($r = 2.0, k = 0.3, d_1 = 0.1, a = 0.02, c = 1.0, m = 0.8, e = 0.7, d_2 = 0.25, q_1 = 0.1, q_2 = 0.1, E_1 = 0.1, E_2 = 0.1$) from initial state ($x_0$, y_0) = (1.2, 0.8). Under these values, the interior coexistence equilibrium is $P^* \approx (0.5285, 1.7094)$, with $J_{11}^* \approx 0.3464, Det(J^*) \approx 0.3015$, and critical Hopf threshold $\alpha^* \approx 0.7957$.

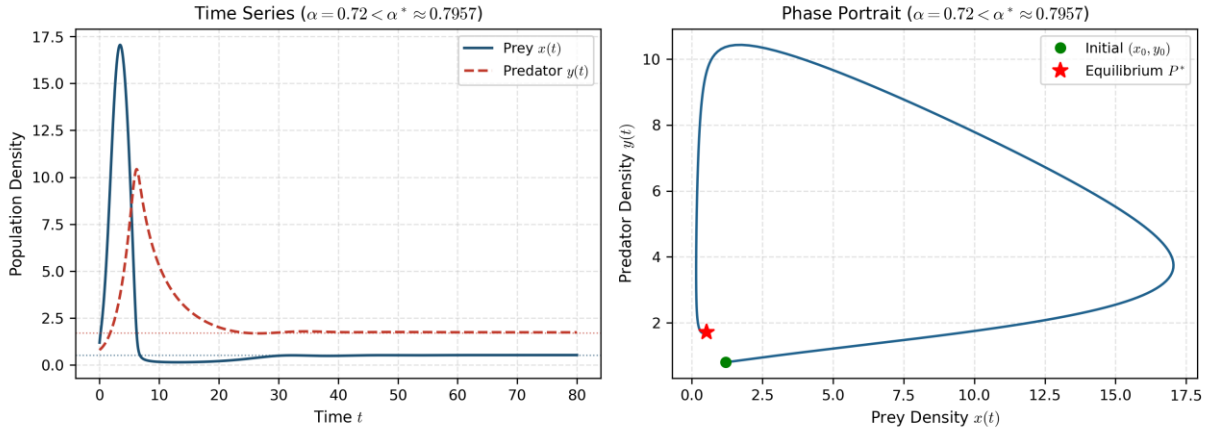


Figure 1: *Time series (left) and phase portrait (right) showing convergence to* $P^* \approx (0.5285, 1.7094)$ for $\alpha = 0.72 < \alpha^*$.

As shown in Figure 1, for $\alpha = 0.72 < \alpha^*$, oscillations decay and trajectories converge to P^*, confirming Theorem 2. When α increases to $\alpha = 0.92 > \alpha^*$ (Figure 2), P^* becomes unstable and trajectories settle onto a periodic limit cycle, confirming Theorem 3.

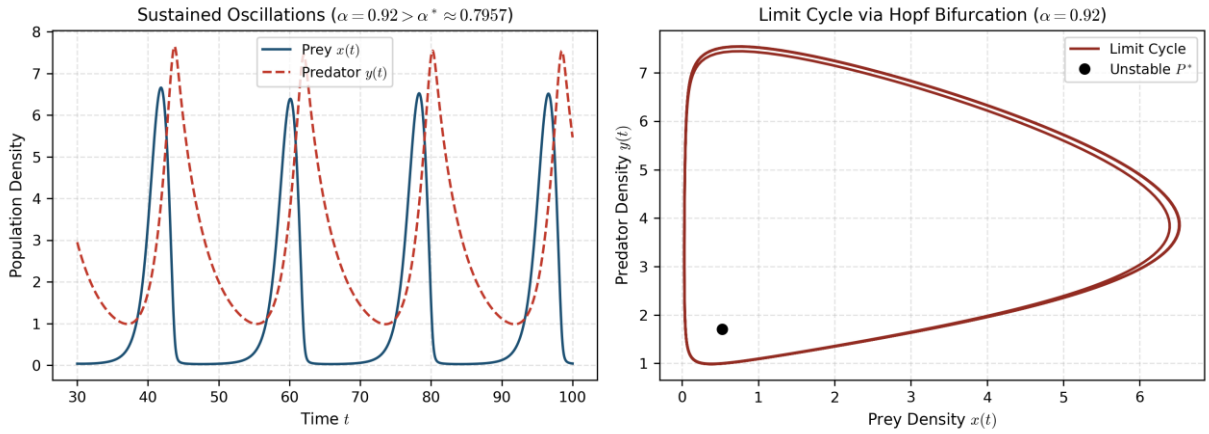


Figure 2: *Sustained periodic oscillations (left) and phase-plane limit cycle (right) at* $\alpha = 0.92 > \alpha^*$.

Figure 3 shows the effect of the fear parameter $k \in \{0.0, 0.3, 1.0, 2.5\}$ at $\alpha = 0.85$. Greater fear levels will decrease both species' peak and equilibrium densities, and dampen large amplitude fluctuations, while causing prey reproduction to be suppressed. The result of comparison of the population dynamics between baseline constant harvesting ($E_1 = E_2 = 0.1$) and biomass-aware harvesting policy bounded in [0,0.35] is shown in Figure 4, where the population dynamics are smoother and the exploitation pressure is also reduced during the low density phases.

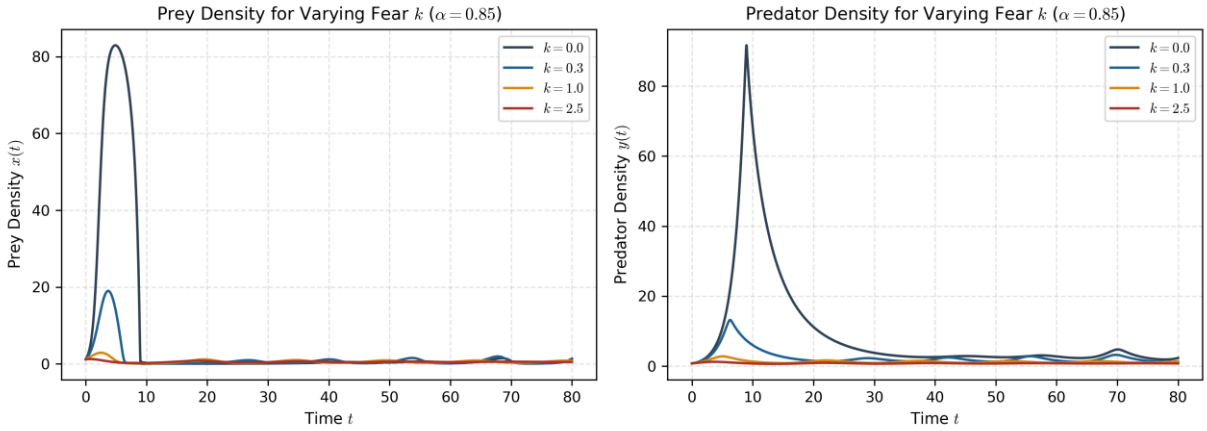


Figure 3: *Prey density (left) and predator density (right) for* $k \in \{0.0, 0.3, 1.0, 2.5\}$ *at* $\alpha = 0.85$.

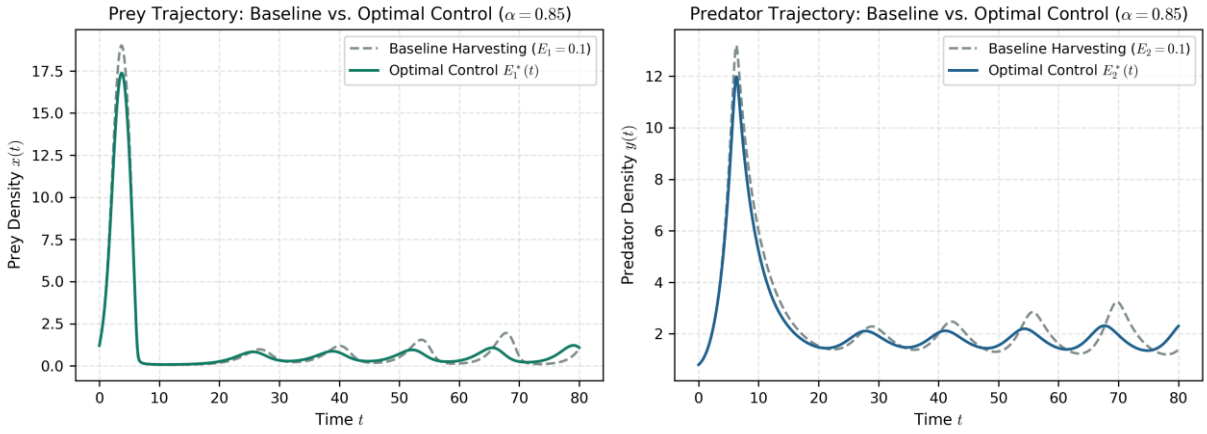


Figure 4: Trajectories of prey (left panel) and predator (right panel) for baseline harvesting and optimal harvesting control with $\alpha = 0.85$.

Table 2: Summary of numerical simulation regimes for system (2)

Scenario / Parameter	Fractional Order α	Observed Dynamical Behavior
Sub-threshold memory	$\alpha = 0.72$ ($< \alpha^* \approx 0.7957$)	Damped oscillations; convergence to $P^* \approx (0.5285, 1.7094)$
Post-bifurcation memory	$\alpha = 0.92$ ($> \alpha^* \approx 0.7957$)	Sustained oscillations; periodic limit cycle around P^*
Fear variation (k = 0.0 to 2.5)	$\alpha = 0.85$	Lower prey/predator peaks and moderated oscillations as k rises
Optimal harvesting control	$\alpha = 0.85$	Reduced amplitude swings and sustainable biomass regulation

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7. Conclusions and Future Directions

We studied the Caputo fractional-order predator-prey model with prey reproductive fear, Holling type-II predation and linear harvesting. Well-posedness (existence, uniqueness, non-negativity and uniform boundedness) were established in $\mathbb{R}^2_+$. The derivation of stability conditions according to Matignon's criterion and an explicit Hopf bifurcation threshold α^*accompanied by an explicit harvesting optimization formula were given and checked numerically by using the Adams-Bashforth-Moulton method. Time delays, spatial diffusion and stochastic perturbations are planned for future extensions.

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